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Robust Distributed $\mathcal{H}_\infty$ Control of Electrical Power Systems

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Abstract— **We consider the problem of synthesizing an optimal robust distributed controller for real-time power balance control in large-scale electrical power systems. Various sufficient robust performance analysis results are summarized together with a computationally tractable distributed controller synthesis algorithm. The proposed synthesis algorithm is tested on a benchmark example of a multi-area power system. The closed loop performance with obtained robust distributed controller is compared with performance of the optimal centralized $\mathcal{H}_\infty$ controller.**

I. Introduction

Major paradigm shifts are taking place in the area of control of electrical power nets. The liberalization of the electricity power market has caused a shift from monopolistic centralized solutions on questions of capacity planning and control to decentralized ones. The substantial increase of distributed and renewable power generators (wind turbines, photovoltaic cells, etc) contribute to power generation but not to frequency stabilization and robustness of the net. Also, the transmission of power has shifted from unidirectional to multi-directional structures in the net. Finally, fluctuations in demand and supply together with capacity disturbances have caused major risks on the stable operation of the net.

The current control structures in which primary controllers on individual generators are combined with Automatic Generation Controllers (AGC's) as second layers to monitor grid frequency deviations and tie-line power fluctuations in specific control areas, falls short in providing guarantees on the robust operation of the power net. One reason for this is the lack of communication between control strategies of neighboring control areas. It is for this reason that an investigation of distributed control architectures for grid frequency and tie-line power stabilization of power nets is of crucial importance.

This paper contributes with a novel algorithm for the complete synthesis of a robust distributed H_∞ controller architecture using an LMI approach. The algorithm is applicable to any graph of interconnected linear time-invariant dynamical systems with uncertainties represented through linear fractional representations. The results of this paper build on earlier contributions by [1], [2], [3]. We extend and generalize these works towards a novel algorithm that explicitly takes uncertainty of the systems into account and that provides explicit stability and robustness guarantees for the synthesized distributed controller.

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A. Notation and preliminaries

The set and field of real numbers is denoted by $\mathbb{R}$, the set of natural numbers by $\mathbb{N}$, the set of $n \times m$ real matrices by $\mathbb{R}^{n\times m}$ and the set of $n \times n$ real symmetric matrices by $\mathbb{S}^n$. The cardinality of a finite set $\mathcal{V}$ is denoted by $|\mathcal{V}|$. The inertia $\mathrm{in}(M)$ of a matrix $M \in \mathbb{S}^n$ is defined as the triplet (a_-, a_0, a_+) of negative, zero and positive eigenvalues of M, respectively. The set $\mathcal{L}_2^n[0,\infty) = \mathcal{L}_2^n$ consists of all measurable functions $f : [0,\infty) \to \mathbb{R}^n$ which satisfy $||f||_2^2 = \int_0^\infty |f(t)|^2 \mathrm{d}t < \infty$. For $T > 0$, the function $f_T : [0,\infty) \to \mathbb{R}^n$ is defined as $f_T(t) = f(t)$ for $0 \le t < T$ and $f_T(t) = 0$ otherwise. The extended set $\mathcal{L}_{2e}^n[0,\infty) = \mathcal{L}_{2e}^n$ consists of all measurable functions $f : [0,\infty) \to \mathbb{R}^n$ such that $f_T \in \mathcal{L}_2^n$ for all $T \in [0,\infty)$. The induced gain of the operator $F : \mathcal{L}_{2e}^m \to \mathcal{L}_{2e}^n$ is given by $||F||_{2,2} = \sup_{f\in\mathcal{L}_2^n,\ f\neq 0} \frac{||F(f)||_2}{||f||_2}$. If F is a Linear Time Invariant (LTI) operator, this induced matrix norm is equal to $||F||_\infty$.

Let Σ be a time-invariant dynamical system described by

$$\dot{x} = f(x,w), \qquad w \in W, \tag{1a}$$
$$z = g(x,u), \qquad z \in Z, \tag{1b}$$

where x is the state, taking values in a state space X, and W and Z are linear spaces. We assume that the system is causal and well-posed. Let $s : W \times Z \to \mathbb{R}$ be a function defined on the space of external variables, and assume that for all $t_0, t_1 \in \mathbb{R}$ and for all input-output pairs (w,z) satisfying (1) the composite function $s(w(t), z(t))$ is locally absolutely integrable, i.e. $\int_{t_0}^{t_1} s(w(t), z(t))\mathrm{d}t < \infty$. The mapping s will be referred to as the *supply function*.

Definition 1 *The system Σ is* dissipative *with respect to the supply function s if there exists a* storage function $V : X \to \mathbb{R}$ *such that*

$$V\big(x(t_0)\big) + \int_{t_0}^{t_1} s\big(u(t), y(t)\big)dt \ge V\big(x(t_1)\big) \tag{2}$$

for any $t_0 \le t_1$ and all signals (w,x,z) which satisfy (1) *with $x(t_0) = x_0$ for any $x_0 \in X$.*

A state space system Σ with $W = \mathbb{R}^m$, $Z = \mathbb{R}^p$ has induced $\mathcal{L}_2$-gain smaller or equal to γ if it is dissipative with respect to the supply function $s(w,z) = \gamma\|w\|^2 - \frac{1}{\gamma}\|z\|^2$.

II. System description

A. Control area model description

For real-time power balance control purposes, large-scale power systems are divided in control areas. A control area can correspond to a country (what is often the case in Europe), but is in general defined as the part of the power system that is capable and responsible for controlling its own power balance in real-time. It is common to model all the generators in a control area with one composite generating unit which sufficiently well approximates the composite behavior of the different generators in the area. A schematic representation of the standard model of a control area is presented in Figure 1.

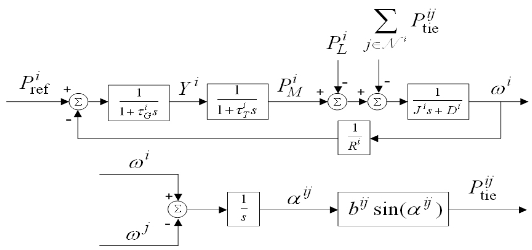

Fig. 1: The model of a control area and a tie-line

1) Model and control objectives: The model of a control area i consists of a turbine with time constant τ_T^i, a governor system with time constant τ_G^i, and has a proportional primary control implemented with as effective speed droop of R^i. The network dynamics in the control area are modeled with inertia J^i and damping D^i. The generating unit has a reference set point P_{ref}^i, and the grid in the control area experiences an exogenous load disturbance P_L^i. The control area is coupled with the other control areas with connecting power lines called tie-lines. The power transfer through the tie-lines is proportional to $\sin(\alpha_{ij})$, where α_{ij} denotes the phase angle difference between the composite generating units i and j, and is denoted by $P_{\text{tie}^{ij}}$. Tie-line power of connecting tie-lines is coupled into the control area at the same node as where the exogenous load disturbance P_L^i is entering the model. The governing system equations are summarized in (3) and (4), while for more details the interested reader is referred to [4].

The control area dynamics are given by

$$\dot{Y}^i = \frac{1}{\tau_G^i}(P_{\text{ref}}^i - \frac{1}{R^i}\omega^i - Y^i), \tag{3a}$$

$$\dot{P}_M^i = \frac{1}{\tau_T^i}\left(Y^i - P_M^i\right), \tag{3b}$$

$$\dot{\omega}^i = \frac{1}{J^i}(P_M^i - P_L^i - D^i\omega^i - \sum_{j\in\mathcal{N}^i} P_{\text{tie}}^{ij}), \tag{3c}$$

where $\mathcal{N}^i$ denotes the index set of areas adjacent to area i. The tie-line relations are given by

$$P_{\text{tie}}^{ij} = b^{ij}\sin(\alpha^{ij}), \tag{4a}$$

$$\dot{\alpha}^{ij} = \omega^i - \omega^j, \tag{4b}$$

$$\alpha^{ji} = -\alpha^{ij}. \tag{4c}$$

All the signals should be regarded as deviations around a certain setpoint. One of the control objectives of the overall system is to bring the overall tie-line power flows among control areas to their scheduled values. Moreover, the control area frequency deviations ω^i should be brought back to zero as quick as possible. These are the crucial objectives of any real-time power balancing control scheme.

2) Uncertainties: Power systems are large-scale systems composed of many interconnected subsystems, each of which usually exhibits nonlinear dynamics and is characterized by various types of uncertainties. For example, it is difficult to determine accurate values of the control area damping parameter D^i. To a large extend, this parameter depends on time-varying characteristics of the loads connected to the control area. Also, the inertia J^i of control area i is subject to parametric uncertainty. Furthermore, relatively low order linear models are suitable for controller synthesis, but neglect dynamical features of the real system, including various types of nonlinearities. One example of such a nonlinearity is the $\sin$ function in (4a).

As in standard robust control, we represent an uncertain subsystem G_Δ^i by pulling out uncertain, non-linear or time-varying elements from the nominal system dynamics, as illustrated in Figure 4. This yields a representation consisting of the interconnection of a nominal LTI system G_0^i and a causal operator Δ^i which represents the uncertainty [5],[6].

B. Generic model of an uncertain power system

In this subsection we present a model of an uncertain power system in a suitable generic form.

Consider a graph $\mathcal{G}_{G_\Delta} = (\mathcal{V}_{G_\Delta}, \mathcal{E}_{G_\Delta})$ in which the set of vertices $\mathcal{V}_{G_\Delta}$ is identified with the set of uncertain subsystems $\{G_\Delta^1, \ldots, G_\Delta^L\}$. The set of nonoriented edges $\mathcal{E}_{G_\Delta} \subseteq \mathcal{V}_{G_\Delta} \times \mathcal{V}_{G_\Delta}$ is defined as follows: $(G_\Delta^i, G_\Delta^j) \in \mathcal{E}_{G_\Delta}$ if subsystems G_Δ^i and G_Δ^j are directly interconnected. For the considered power system application, control area i and control area j are directly interconnected if there is a tie-line between them, i.e. when the dynamics of the area i is directly influenced by the dynamics of the adjacent area j via (3c).

For all $i = 1, \ldots, L$, the dynamics of the uncertain LTI system G_Δ^i is represented by

$$p^i(t) = \Delta^i(q^i(t)) \qquad \forall\, \Delta^i \in \boldsymbol{\Delta}^i \tag{5}$$

$$\begin{bmatrix} \dot{x}^i(t) \\ w^i(t) \\ q^i(t) \\ z^i(t) \\ y^i(t) \end{bmatrix} = \begin{bmatrix} A_{TT}^i & A_{TS}^i & B_{Tp}^i & B_{Td}^i & B_{Tu}^i \\ A_{ST}^i & A_{SS}^i & B_{Sp}^i & B_{Sd}^i & B_{Su}^i \\ C_{Tq}^i & C_{Sq}^i & D_{qp}^i & D_{qd}^i & D_{qu}^i \\ C_{Tz}^i & C_{Sz}^i & D_{zp}^i & D_{zd}^i & D_{zu}^i \\ C_{Ty}^i & C_{Sy}^i & D_{yp}^i & D_{yd}^i & D_{yu}^i \end{bmatrix} \begin{bmatrix} x^i(t) \\ v^i(t) \\ p^i(t) \\ d^i(t) \\ u^i(t) \end{bmatrix} \tag{6}$$

where $(d^i, z^i) \in \mathbb{R}^{n_d^i + n_z^i}$ is the *performance channel*, $(v^i, w^i) \in \mathbb{R}^{2n_G^i}$ is the *interconnection channel*, $(p^i, q^i) \in \mathbb{R}^{2n_\Delta^i}$ is the *uncertainty channel*, and $(u^i, y^i) \in \mathbb{R}^{n_u^i + n_y^i}$ is the *control channel*. Furthermore, $x^i(t) \in \mathbb{R}^{m^i}$ denotes the state variable and the subscripts T, S in (6) denote temporal (states) and spatial (interconnections) dynamics, respectively. In (5), $\boldsymbol{\Delta}^i$ is a set of causal operators on $\mathcal{L}_{2e}^{n_\Delta^i}$ with bounded gain. We denote the nominal state space representation (6) as G_0^i. Whenever $(G_\Delta^i, G_\Delta^j) \in \mathcal{E}_{G_\Delta}$, the interconnection channel (w^i, v^i) of subsystem i is further partitioned such that $(w^{ij}, v^{ij}) \in \mathbb{R}^{2n_G^{ij}}$ denotes the interconnection channel between subsystem i and j. By constraining w^{ij} and v^{ij} to share the same dimension n_G^{ij}, the spatial dynamics matrix

A_{SS}^i will be square, simplifying the analysis later on. Note that this can always be done by adding zero rows or columns to A_{SS}^i.

The signals over the interconnection channel (v, w) are restricted to satisfy $w^{ij}(t) = v^{ji}(t)$ and $v^{ij}(t) = w^{ji}(t)$ for all $i > j$, $t \geq 0$, meaning that any interconnection signal w^{ij} which leaves subsystem i arrives directly at subsystem j.

Example 1. Consider a power system system which consists of four connected control areas coupled in a row-like fashion, as presented in Figure 2. We will use this power

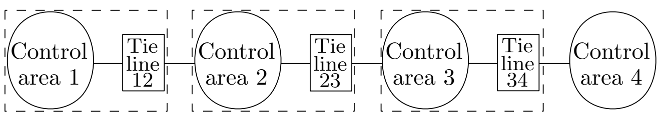

Fig. 2: Four generator model. The tie line dynamics between area i and j are lumped into the ith area dynamics for $i < j$.

system topology in case studies presented in Section V. When implementing the model for control purposes, it is

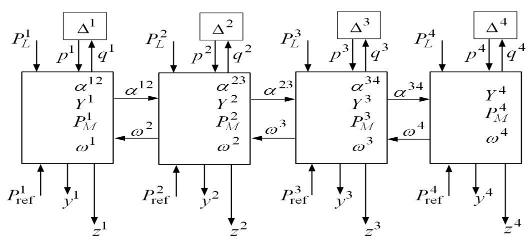

Fig. 3: The four-area model.

convenient to incorporate the tie-line dynamics (4) into the four subsystems, as presented in Figure 2 and Figure 3. The sensor measurements y^i in the control channel are the *local* tie-line power flows and the *local* network frequency deviation ω^i. These signals are the input to the controller of the corresponding area. The controller output u^i is the reference setpoint P_{ref}^i. The performance signal z^i consists of both y^i and u^i. For example, in the case of subsystem 2 in Figure 3, the signals are defined as $y^2 = \text{col}(P_{\text{tie}}^{12}, P_{\text{tie}}^{23}, \omega^2)$, $z^2 = \text{col}(y^2, u^2) = \text{col}(P_{\text{tie}}^{12}, P_{\text{tie}}^{23}, \omega^2, P_{\text{ref}}^2)$, $v^{21} = \alpha^{12}$, $w^{21} = \omega^2$, $v^{23} = \omega^3$, $w^{23} = \alpha^{23}$, $v^2 = \text{col}(v^{21}, v^{23})$, $w^2 = \text{col}(w^{21}, w^{23})$. □

III. Problem formulation

A. Control configuration

Given an uncertain distributed system $\mathcal{G}_{G_\Delta}$ with $|\mathcal{V}|_{G_\Delta} = L$. A distributed controller for the system is a graph $\mathcal{G}_K = (\mathcal{V}_K, \mathcal{E}_K)$, $|\mathcal{V}_K| = L$ where we identify with $\mathcal{V}_K$ the set $\{K^1, \ldots, K^L\}$ of local controllers

$$\begin{bmatrix} \dot{x}_K^i(t) \\ w_K^i(t) \\ u^i(t) \end{bmatrix} = \begin{bmatrix} (A_{TT}^i)_K & (A_{TS}^i)_K & (B_T^i)_K \\ (A_{ST}^i)_K & (A_{SS}^i)_K & (B_S^i)_K \\ (C_{Tq}^i)_K & (C_{Sq}^i)_K & D_K^i \end{bmatrix} \begin{bmatrix} x_K^i(t) \\ v_K^i(t) \\ y^i(t) \end{bmatrix},$$

and where $(K^i, K^j) \in \mathcal{E}_K$ if and only if $(G_\Delta^i, G_\Delta^j) \in \mathcal{E}_{G_\Delta}$. Here, $x_K^i \in \mathbb{R}^{m^i}$ denotes the state variable, $(v_K^i, w_K^i) \in \mathbb{R}^{2n_K^i}$ is the controller interconnection channel, and $(u^i, y^i) \in \mathbb{R}^{n_u^i + n_y^i}$ is the control channel. The controller interconnection channel is further partitioned into $(v_K^{ij}, w_K^{ij}) \in n_K^{ij}$ for any pair i, j such that $(K^i, K^j) \in \mathcal{E}_K$, equivalently to the partitioned interconnection channels in the subsystems of $\mathcal{G}_{G_\Delta}$.

An uncertain distributed system and a controller define the *controlled system* $\mathcal{G}_C = (\mathcal{V}_C, \mathcal{E}_C)$ where $\mathcal{V}_C := \{(G_\Delta^i)_C := S(G_\Delta^i, K^i), i = 1, \ldots, L\}$ with S defining the interconnection of G_Δ^i with the controller K^i over the control channel. It follows that $((G_\Delta^i)_C, (G_\Delta^j)_C) \in \mathcal{E}_C$ if and only if $(G_\Delta^i, G_\Delta^j) \in \mathcal{E}_G$. All interconnection variables are illustrated in Figure 4.

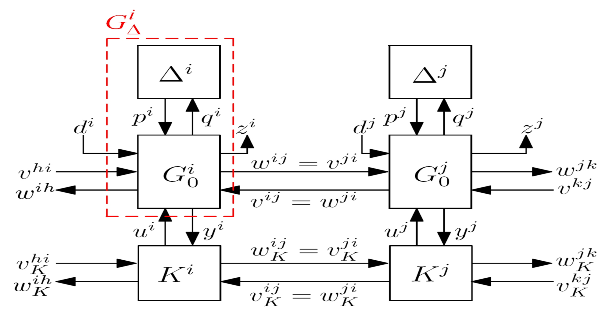

Fig. 4: Interconnected uncertain subsystems with a distributed controller.

B. Control objectives

The graph of the controlled system $\mathcal{G}_C$ defines a causal operator $(G_\Delta)_C$ on $\mathcal{L}_{2e}^{n_d}$ such that

$$(G_\Delta)_C: \ d := \text{col}(d^1, \ldots, d^L) \to z := \text{col}(z^1, \ldots, z^L).$$

Robust performance of the controlled system is now defined as follows.

Definition 2 *We say that the controlled system $\mathcal{G}_C$ achieves* robust performance *of level γ if it is well-posed, robustly stable and with initial condition $(x^i(0), x_K^i(0)) = (0, 0)$ it satisfies*

$$||\,(G_\Delta)_C\,||_{2,2} = \sup_{d \neq 0, d \in \mathcal{L}_2^{n_d}, \Delta \in \mathbf{\Delta}} \frac{||z||_2}{||d||_2} < \gamma \qquad (7)$$

The formal problem definition is now the following.

Problem 1 *Given an uncertain distributed system $\mathcal{G}_{G_\Delta}$ defined as in Section II-B, synthesize a distributed controller $\mathcal{G}_K$ with structure as given in Section III-A such that the resulting controlled uncertain distributed system $\mathcal{G}_C$ achieves robust performance of level γ.*

IV. Synthesis algorithm

In this section the closed-loop system robust performance analysis results are presented in terms of nonlinear matrix inequalities, together with an efficient and constructive algorithm for controller synthesis as a solution to Problem 1. Detailed descriptions and all the proofs are reported in [7].

A. Closed loop analysis

Suppose that we are given a distributed controller as described in Section III-A. We denote the local interconnection of subsystem and controller at node i as $\left(G^i_\Delta\right)_C$, which admits the following representation

$$p^i = \Delta^i(q^i) \qquad \forall\, \Delta^i \in \mathbf{\Delta}^i \tag{8}$$

$$\begin{bmatrix} \dot{x}^i_C \\ v^i_C \\ p^i \\ z^i \end{bmatrix} = \begin{bmatrix} (A^i_{TT})_C & (A^i_{TS})_C & (B^i_{Tp})_C & (B^i_{Td})_C \\ (A^i_{ST})_C & (A^i_{SS})_C & (B^i_{Sp})_C & (B^i_{Sd})_C \\ (C^i_{Tq})_C & (C^i_{Sq})_C & (D^i_{qp})_C & (D^i_{qd})_C \\ (C^i_{Tz})_C & (C^i_{Sz})_C & (D^i_{zp})_C & (D^i_{zd})_C \end{bmatrix} \begin{bmatrix} x^i_C \\ w^i_C \\ q^i \\ d^i \end{bmatrix} \tag{9}$$

where the closed loop states and interconnection signals are now appended versions of the open loop ones. Hence, $x^i_C = \mathrm{col}(x^i, x^i_K) \in \mathbb{R}^{2m^i}$, $v^i_C = \mathrm{col}(v^i, v^i_K) \in \mathbb{R}^{n^i_C}$ and $w^i_C = \mathrm{col}(w^i, w^i_K) \in \mathbb{R}^{n^i_C}$. We denote the state space realization in (9) as $\left(G^i_0\right)_C$.

Theorem 1 *[7] Let $\mathcal{G}_{(G_\Delta)_C}$ be an uncertain distributed system with subsystems admitting realizations* (8)-(9). *Then, the system is well-posed, stable and achieves robust performance γ if for all i there exist symmetric matrices $X^i_C \in \mathbb{S}^{2m}$, $Z^i_C \in \mathbb{S}^{n^i_C}$ and $D^i_\Delta \in \mathbb{S}^{2n^i_\Delta}$, such that $X^i_C \succ 0$, $D^i_{11} \succ 0$, $D^i_{22} \prec 0$ and for all $\Delta^i \in \mathbf{\Delta}^i$ and $q^i \in \mathbb{R}^{n^i_\Delta}$*

$$(T^i)^*_C M^i_C T^i_C \prec 0, \tag{10}$$

$$\begin{bmatrix} q^i \\ \Delta^i(q^i) \end{bmatrix}^* \begin{bmatrix} D^i_{11} & D^i_{12} \\ (D^i_{12})^* & D^i_{22} \end{bmatrix} \begin{bmatrix} q^i \\ \Delta^i(q^i) \end{bmatrix} \succeq 0, \tag{11}$$

where

$$T^i_C := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ \left(A^i_{TT}\right)_C & \left(A^i_{TS}\right)_C & \left(B^i_{Tp}\right)_C & \left(B^i_{Td}\right)_C \\ \hline \left(A^i_{ST}\right)_C & \left(A^i_{SS}\right)_C & \left(B^i_{Sp}\right)_C & \left(B^i_{Sd}\right)_C \\ 0 & I & 0 & 0 \\ \hline \left(C^i_{Tq}\right)_C & \left(C^i_{Sq}\right)_C & \left(D^i_{qp}\right)_C & \left(D^i_{qd}\right)_C \\ 0 & 0 & I & 0 \\ \hline \left(C^i_{Tz}\right)_C & \left(C^i_{Sz}\right)_C & \left(D^i_{zp}\right)_C & \left(D^i_{zd}\right)_C \\ 0 & 0 & 0 & I \end{array}\right],$$

$$M^i_C := \mathbf{diag}\left(\begin{bmatrix} 0 & \left(X^i_T\right)_C \\ \left(X^i_T\right)_C & 0 \end{bmatrix}, \begin{bmatrix} \left(Z^i_{11}\right)_C & \left(Z^i_{12}\right)_C \\ \left(Z^i_{12}\right)^*_C & \left(Z^i_{22}\right)_C \end{bmatrix}, \begin{bmatrix} D^i_{11} & D^i_{12} \\ \left(D^i_{12}\right)^* & D^i_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{bmatrix}, \right)$$

with $\left(Z^i_{11}\right)_C$ partitioned to $\begin{bmatrix} \left(Z^i_{11}\right)_G & \left(Z^i_{11}\right)_{GK} \\ \left(Z^i_{11}\right)^*_{GK} & \left(Z^i_{11}\right)_K \end{bmatrix}$, and $\left(Z^i_{12}\right)_C, \left(Z^i_{22}\right)_C$ and $\left(X^i_T\right)_C$ defined analogously.

Because of space limitations we cannot provide a formal proof of this result here. The main observation of Theorem 1 is that it concludes robust performance of the global distributed system from dissipation properties of its constituent subsystems. As such, the result is similar to the standard strict dissipativity result for linear systems as shown in [8], combined with neutrality and robustness properties of interconnected systems. Basically, Theorem 1 promises that the interconnected system is strictly dissipative with quadratic supply function $\gamma\|d\|^2 - \frac{1}{\gamma}\|z\|^2$ whenever each nominal subsystem G^i_0 is strictly dissipative with respect to the supply function $p_i + s_i - u_i$ where $p_i(d^i, z^i) = \gamma\|d^i\|^2 - \frac{1}{\gamma}\|z^i\|^2$ is the *local performance supply*, $s_i(v^i_C, w^i_C) = \sum_j \mathrm{col}(v^{ij}_C, w^{ij}_C)^\top X^{ij}\,\mathrm{col}(v^{ij}_C, w^{ij}_C)$ is the *local interconnection supply* function and $u_i(p^i, q^i) = \mathrm{col}(q^i, p^i)^\top D_i\,\mathrm{col}(q^i, p^i)$ is the *local uncertainty supply* function. Here, the design scales X^{ij} and D_i can be chosen freely as long as they satisfy the inequalities (10) and (11) which, in fact, imply a neutrality condition $\sum_{i=1}^L P_i = 0$ over the interconnection channels and a nonnegativity condition $\sum_{i=1}^L U_i \geq 0$ over the uncertainty channels. This local dissipation property admits a characterization in terms of coupled linear matrix inequalities as is given in Theorem 1.

B. Synthesis inequalities

The analysis inequalities of Theorem 1 are not directly suitable for efficient controller synthesis. This is due to the multiplication of unknown matrices M^i with the unknown controller parameters that are present in T^i_C, i.e., the inequalities are non-linear matrix inequalities if the controller is unknown. Using the so-called elimination lemma [9], [1], [8], one can eliminate the controller parameters from the inequalities in Theorem 1, as follows.

Theorem 2 *[7] There exist matrices such that the conditions of Theorem 1 are satisfied with $n^{ij}_K = 3n^{ij}_G$ if and only if there exist symmetric matrices $M^i, \tilde{M}^i \in \mathbb{S}^{2(m^i + n^i_G + n^i_\Delta) + n^i_d + n^i_z}$ such that for all $\Delta^i \in \mathbf{\Delta}^i$, $D^i_{11} \succ 0$, $D^i_{22} \prec 0$, $\left(X^i_T\right)_G, \left(Y^i_T\right)_G \succ 0, q^i \in \mathbb{R}^{n^i_\Delta}$,*

$$(U^i_G)^*_\perp (T^i)^* M^i_G T^i (U^i_G)_\perp \prec 0, \tag{12}$$

$$(V^i_G)^*_\perp (T^i_\perp)^* \tilde{M}^i_G T^i_\perp (V^i_G)_\perp \succ 0, \tag{13}$$

$$\begin{bmatrix} (X_T)^i_G & I \\ I & (Y_T)^i_G \end{bmatrix} \succeq 0, \tag{14}$$

$$\begin{bmatrix} q^i \\ \Delta^i(q^i) \end{bmatrix}^* \begin{bmatrix} D^i_{11} & D^i_{12} \\ (D^i_{12})^* & D^i_{22} \end{bmatrix} \begin{bmatrix} q^i \\ \Delta^i(q^i) \end{bmatrix} \succeq 0. \tag{15}$$

Here,

$$T^i := \left[\begin{array}{cccc} I & 0 & 0 & 0 \\ A^i_{TT} & A^i_{TS} & B^i_{Tp} & B^i_{Td} \\ \hline A^i_{ST} & A^i_{SS} & B^i_{Sp} & B^i_{Sd} \\ 0 & I & 0 & 0 \\ \hline C^i_{Tq} & C^i_{Sq} & D^i_{qp} & D^i_{qd} \\ 0 & 0 & I & 0 \\ \hline C^i_{Tz} & C^i_{Sz} & D^i_{zp} & D^i_{zd} \\ 0 & 0 & 0 & I \end{array}\right],$$

$$M^i_G := \mathbf{diag}\left(\begin{bmatrix} 0 & (X_T)^i_G \\ (X_T)^i_G & 0 \end{bmatrix}, \begin{bmatrix} \left(Z^i_{11}\right)_G & \left(Z^i_{12}\right)_G \\ \left(Z^i_{12}\right)^*_G & \left(Z^i_{22}\right)_G \end{bmatrix}, \begin{bmatrix} D^i_{11} & D^i_{12} \\ \left(D^i_{12}\right)^* & D^i_{22} \end{bmatrix}, \begin{bmatrix} \frac{1}{\gamma} I & 0 \\ 0 & -\gamma I \end{bmatrix} \right),$$

$$\tilde{M}^i_G := \mathbf{diag}\left(\begin{bmatrix} 0 & (Y_T)^i_G \\ (Y_T)^i_G & 0 \end{bmatrix}, \begin{bmatrix} \left(\tilde{Z}^i_{11}\right)_G & \left(\tilde{Z}^i_{12}\right)_G \\ \left(\tilde{Z}^i_{12}\right)^*_G & \left(\tilde{Z}^i_{22}\right)_G \end{bmatrix}, \begin{bmatrix} D^i_{11} & D^i_{12} \\ \left(D^i_{12}\right)^* & D^i_{22} \end{bmatrix}^{-1}, \begin{bmatrix} \gamma I & 0 \\ 0 & -\frac{1}{\gamma} I \end{bmatrix} \right),$$

$(U^i_G)^*_\perp$ spans the nullspace of $[C^i_{Ty}\; C^i_{Sy}\; D^i_{yp}\; D^i_{yd}]$,

$(V^i_G)^*_\perp$ spans the nullspace of $[(B^i_{Tu})^*\; (B^i_{Su})^*\; (D^i_{qu})^*\; (D^i_{zu})^*]$,

$$\left(Z^i_{11}\right)_G := -\mathop{\mathbf{diag}}_{j \in \mathcal{N}^i} \left(X^{ij}_{11}\right)_G, \quad \left(\tilde{Z}^i_{11}\right)_G := -\mathop{\mathbf{diag}}_{j \in \mathcal{N}^i} \left(Y^{ij}_{11}\right)_G,$$

and $\left(Z^i_{12}\right)_G$, $\left(Z^i_{22}\right)_G$, $\left(\tilde{Z}^i_{12}\right)_G$, $\left(\tilde{Z}^i_{22}\right)_G$ are defined analogously.

The result of Theorem 2 still does not define a convex optimization in all variables. Applying the elimination lemma on the analysis equations in Theorem 1 removed the controller parameters from the equations, thus removing the bilinearity obtained from multiplying controller parameters with the variables in the matrix M_C^i. However, equations (12) and (13) give a set of LMIs in both D_Δ^i and $\left(D_\Delta^i\right)^{-1}$ which renders the combined set of equations non-convex. No convexifying operation is known to date in the LMI framework, and in general the problem of finding a robust performance bound even for non-distributed systems is known to be NP-hard [10]. An algorithm for solving the robust distributed control problem is suggested in the following subsection.

C. Controller synthesis algorithm

Theorem 2 solves the problem of finding a controller, given some non-singular D_Δ^i which represents the uncertainty. Theorem 1 on the other hand finds, given a certain controller, the D_Δ^i which represents the uncertainty in such a way that a minimal γ is obtained. Therefore, the following iterative synthesis algorithm can be used, which is similar to the DK-iterative type algorithms from μ synthesis.

Algorithm 1

1) ***Initialization*** *$j = 0$, tolerance $\epsilon > 0$ and maximum iteration index $j_{max} \in \mathbb{N}$;*
   a) *Initial K: Solve* (12)-(15) *for $\mathcal{G}_{G_0}$, hence $\boldsymbol{\Delta_c^i} = 0 \forall\ i$. Construct initial $\mathcal{G}_{K_0}$ by reconstruction as described in [1].*
   b) *Use nominal controller to minimize γ subject to feasibility of* (10)-(11)*, thus obtaining some $\gamma_{0,D}$ and $\left(D_\Delta^i\right)_0$ for all i such that there is robust stability and performance bounded by $\gamma_{0,D}$. Update j to $j = 1$.*
2) ***K-step*** *Plug in $\left(D_\Delta^i\right)_{j-1}$ and minimize γ subject to feasibility of* (12)-(15) *to obtain a new performance bound $\gamma_{j,K}$ and, after reconstruction, a new $\mathcal{G}_{K_j}$.*
3) ***D-step*** *Generate a closed loop system by plugging in controller $\mathcal{G}_{K_j}$ and minimize γ subject to feasibility of* (10)-(11) *to find a new $\left(D_\Delta^i\right)_j$ and performance bound $\gamma_{j,D}$. Update $j = j + 1$.*
4) ***Termination*** *Terminate when $\gamma_{j,K} - \gamma_{j-1,K} < \epsilon$ or when $j = j_{max}$. Else, return to step 2.*

Note that the above described algorithm does not guarantee convergence to a global optimum, thus leading to possibly conservative solutions. Some further properties of the algorithm are summarized as follows.

Theorem 3 *[7] Consider an uncertain distributed system $\mathcal{G}_{G_\Delta}$. Suppose that there exist a $\mathcal{G}_{K_0}, \gamma_{0,D}$ and $\left(D_\Delta^i\right)_0$ for all i as defined in step 1 of Algorithm 1. Then, the solution to Algorithm 1 has the following properties:*

1) *$\gamma_{j,K} \leq \gamma_{j-1,K}$ for all $j \leq j_{max}, j \in \mathbb{N}$*
2) *Problem 1 is solved with suboptimal robust performance bound $\gamma = \lim_{j\to\infty} \gamma_{j,K}$ approximated by $\gamma_{j_{max},K}$*

V. Case study

We have implemented the distributed controller synthesis algorithm on the benchmark power system presented in Example 1, for which all the corresponding system parameters for (3), (4) can be found in [11].

A. Nominal system simulations

Performance of the nominal distributed controller is compared to a centralized controller which internally stabilizes the system and minimizes the $\mathcal{H}_\infty$-norm of the closed-loop transfer functions $d \to z$, as well as to the decentralized Automated Gain Control (AGC) controllers currently used in power networks [4]. The setup is tested using a stepwise load increase of 25% in area 2 at time $t = 10s$ and a 25% load decrease in area 3 at $t = 20$s. The time-domain results are presented in Figure 5. We can observe that the distributed

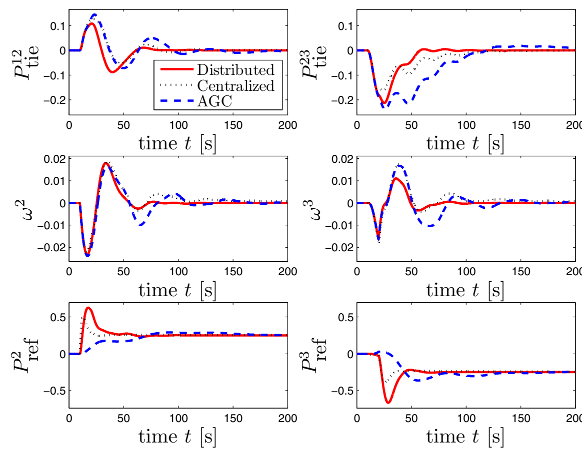

Fig. 5: Comparison of time domain behavior of different control strategies on the nominal system.

controller obtains good performance as compared to both its centralized counterpart and the traditional AGC controller. The distributed controller is faster in regulating frequency deviation and tie line power back to zero with about equal control effort. The resulting singular value plots are shown in Figure 6. A comparison of infinity norms of the resulting closed loop systems is presented in the first column of Table I. Note that the presented values are given for the closed loop system where the filters used in the $\mathcal{H}_\infty$ shaping design are removed. We see that the presented distributed controller obtains comparable performance level to the centralized controller, and outperforms the AGC configuration.

Controller	$\lVert(G_0)_C\rVert_\infty$	$\lVert(G_\Delta)_C\rVert_\infty$
Nominal Centralized $\mathcal{H}_\infty$	5.40	4.70
Nominal Distributed $\mathcal{G}_C$	4.58	23.03
Robust Distributed $\mathcal{G}_C$	-	523.56
Automated Gain Controller	5.06	-

TABLE I: Performance bounds of the resulting closed loop systems.

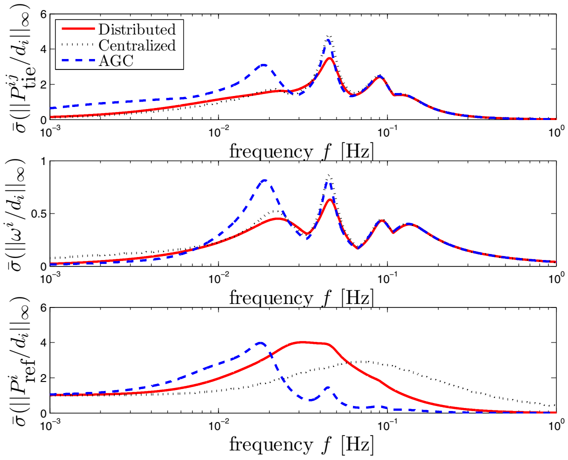

Fig. 6: Comparison of the maximum singular values with different control strategies, per output channel.

B. Uncertainty modeling

We consider the tie line power P_{tie}^{ij} as a cause of uncertainty in the system, since a modeling error is made by approximating the sinusoid relation in (4) as a linear relation. Taking this uncertainty into account, we assume a sector bound uncertainty model [6] for $-\frac{\pi}{2} < \alpha^{ij} < \frac{\pi}{2}$. The tie line dynamics are thus modeled by the parametric uncertainty: $P_{\text{tie}}^{ij} = b^{ij}\delta^{ij}\alpha^{ij}$; $\frac{2}{\pi} \leq \delta^{ij} \leq 1$ Furthermore, to show performance of the algorithm, we also introduce an uncertainty of 50% in the inertia of plant 1 according to $J^1 = \delta_J^1 J_0^1$ with $\delta_J^1 \in [0.5, 1.5]$. This leads to two repeated scalar blocks of size 2 in the uncertainty representation of subsystem 1. The following operators are used: $\Delta^1 = \mathbf{diag}(\delta^{12}I_2, \delta_J^1 I_2)$, $\Delta^2 = \mathbf{diag}(\delta^{12}, \delta^{23})$, $\Delta^3 = \mathbf{diag}(\delta^{23}, \delta^{34})$, $\Delta^4 = \delta^{34}$, $\forall\, \delta^i \in \boldsymbol{\delta}^i, i = 1, \ldots, 4$.

C. Robust simulations

A robust controller of total order $m_K = m_G = 14$ is obtained by Algorithm 1 after approximately 950 seconds and $j = j_{max} = 6$ iterations. The robust distributed controller is applied to a perturbed plant with a specific perturbation lying inside the modeled robustness region. Performance of the robust distributed controller is compared to that of a centralized and nominal distributed controller, both applied to the same perturbed plant. The simulations are shown in Figure 7, with the same external step shaped inputs applied on subsystem 2 and 3 as in the nominal case. The obtained results illustrate the efficiency of concept for robust stability and performance in a distributed setting. The infinity norms of the resulting closed loop systems are presented in column 2 of Table I.

VI. Conclusions

In this paper we have developed an explicit algorithm for the synthesis of robust distributed H_∞ optimal controllers. The algorithm is based on a $D-K$ type of iteration involving feasibility tests on linear matrix inequalities. The algorithm is applicable to an arbitrary graph of interconnected LTI systems and results in a distributed controller with the same graph topology. The distributed controller provides guaranteed stability and H_∞ performance levels in the face of linear fractional representations of plant uncertainties. The efficiency of the proposed synthesis algorithms is illustrated on the benchmark example of electrical power system control.

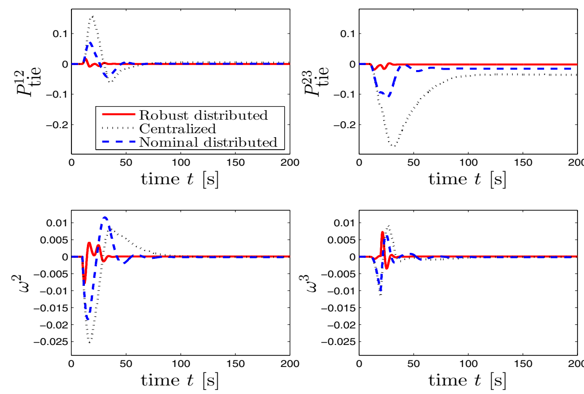

Fig. 7: Comparison of time domain behavior of different control strategies on the perturbed system.

VII. Acknowledgements

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References

[1] C. Langbort, R. S. Chandra, and R. D'Andrea, "Distributed Control Design for Systems Interconnected Over an Arbitrary Graph," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1502–1519, September 2004.

[2] P. Viccione, C. Scherer, and M. Innocenti, "LPV Synthesis with Integral Quadratic Constraints for Distributed Control of Interconnected Systems," Master's thesis, University of Pisa, 2008.

[3] G. Dekker, A. Jokić, and S. Weiland, "Distributed $\mathcal{H}_\infty$-based control of Electrical Power Systems," in *Distributed Estimation and Control in Networked Systems, 2010., Proceedings of the 2nd IFAC Workshop on*, Dec. 2010.

[4] P. Kundur, *Power System Stability and Control*, ser. The EPRI Power System Engineering Series, N. J. Balu and M. G. Lauby, Eds. McGraw-Hill, 1994.

[5] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1996.

[6] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *Automatic Control, IEEE Transactions on*, vol. 42, no. 6, pp. 819 –830, jun 1997.

[7] T. van der Els, A. Jokic, and S. Weiland, "Robust distributed hinfty control," *Technical report*, pp. 1–12, 2011. [Online]. Available: http://w3.ele.tue.nl/nl/onderzoek/publicatielijst-capaciteitsgroepen/cs

[8] C. Scherer and S. Weiland, "Linear matrix inequalities in control," 2005, lecture Notes to DISC (Dutch Institute of Systems anc Control) Course on Linear Matrix Inequalities.

[9] C. Scherer, "LPV control and full block multipliers," *Automatica*, vol. 37, no. 3, pp. 361–375, 2001.

[10] S. Poljak and J. Rohn, "Checking robust nonsingularity is np-hard," *Mathematics of Control, Signals, and Systems (MCSS)*, vol. 6, pp. 1–9, 1993, 10.1007/BF01213466. [Online]. Available: http://dx.doi.org/10.1007/BF01213466

[11] A. Venkat, I. Hiskens, J. Rawlings, and S. Wright, "Distributed MPC strategies with application to power system automatic generation control," *IEEE Transactions on Control Systems Technology*, vol. 16, no. 6, pp. 1192–1206, 2008.